Modular invariance and the Feigin-Fuch representation of characters for $SU_k(2)$ WZW and minimal models

1991 J. Phys. A: Math. Gen. 24 11

(http://iopscience.iop.org/0305-4470/24/1/012)

J. Phys. A: Math. Gen. 24 (1991) 11-21. Printed in the UK

Modular invariance and the Feigin–Fuch representation of characters for $SU_k(2)$ WZW and minimal models

Bo-Yu Hou† and Rui-Hong Yue†‡

† Institute of Modern Physics, Northwest University, Xian 710069, People's Republic of China

‡ CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China

Received 21 May 1990, in final form 18 September 1990

Abstract. In this paper, we have discussed the modular properties of characters in Feigin–Fuch representation for $SU_k(2)$ WZW and minimal models, and we have given the explicit expressions of the S-matrix. The proof is given for a simple case.

1. Introduction

In recent years, a useful method has been developed to discuss the correlation function and quantum group structure for rational conformal field theories (RCFT) [1-5]. In this method, the conformal block can be written as a Feigin-Fuch (FF) contour integral and the correlation function is represented as the product of holomorphic and anti-holomorphic conformal blocks. Based on the monodromy invariance of the correlation function, one can obtain the structure constant (operator product expansion), fusion rule and crossing matrices (fusion and braiding matrices) [3-6].

On the other hand, the classification of conformal field theory is an important problem. Mathur *et al* [7, 8] give an approach to produce in principle all RCFT characters based on the modular invariant differential equation which depends on two integers N, the number of characters, and L, which is proportional to the number of zeroes of the Wronskian determinant of the characters in the interior of moduli space. The characters satisfy an Nth order differential equation. For large values of N it is difficult to solve the equation. Mukhi *et al* [9] propose that the characters for $L=0$ and large N can be written as an FF contour integral, and give some useful results.

In this paper we will discuss the monodromy transformation properties of the FF contour integrals and calculate the modular matrices of characters in FF integral representation for the A-D-E classification of $SU_k(2)$ WZW and minimal models. The organization of this paper is as follows. In section 2 we briefly recall the FF integral representation for characters. In section 3 we generally discuss the monodromy behaviour of FF integrals. The transformation matrices of the characters in the A-D-E classification of the $SU_k(2)$ WZW model are given in section 4. In section 5 we obtain the S-matrix for the minimal model, and give some checks for the identity of the monodromy of FF integrals and the modular property of characters in the appendix.

0305-4470/91/010011+11\$03.50

2. FF contour integral representation for characters

In this section, we will recall the FF integral representation for characters of RCFT with $L=0$.

In RCFT, any partition function has the general form

$$Z(\tau)=\sum_{i,j} N_{ij}\chi_i(\tau)\chi_j(\bar{\tau}) \tag{2.1}$$

where $\chi_i(\tau)$ is the character for the i representation of the chiral algebra A_τ. This character can be thought as a multivalued function on the complex. The variable λ on the complex plan has a power series expansion in half-integer power of the variable $q=\mathrm{e}^{\mathrm{i}2\pi\tau}$ (τ on the torus):

$$\lambda=16q^{1/2}(1-8q^{1/2}+44q+\mathrm{O}(q^{3/2})). \tag{2.2}$$

The corresponding generator of SL(2, 2) of the modular transformation in the term λ are

$$S:\lambda\to 1-\lambda$$
$$T:\lambda\to\frac{\lambda}{\lambda-1}. \tag{2.3}$$

First, we investigate the behaviour of the characters of a conformal field theory with conformal centre c and spectrum h_i, which has N characters. A character χ_i associated with a primary field of conformal weight h_i behaves in the $\lambda\to 0$ limit as

$$\chi_i\to\lambda^{-c/12+2h_i}. \tag{2.4}$$

MPS conjecture is that the characters of RCFT with $L=0$ can be written as an FF integral:

$$\begin{aligned}J_{AB}(\lambda)=(\lambda(1-\lambda))^{\alpha}&\int_0^{\lambda}\prod_{i=1}^{A}\mathrm{d}u_i[u_i(1-u_i)(\lambda-u_i)]^a\int_1^{\infty}\prod_{i=A+1}^{n}\mathrm{d}u_i[u_i(u_i-1)(u_i-\lambda)]^a\\&\times\int_0^{\lambda}\prod_{i=1}^{B}\mathrm{d}v_i[v_i(1-v_i)(\lambda-v_i)]^b\int_1^{\infty}\prod_{i=B+1}^{m}\mathrm{d}v_i[v_i(v_i-1)(v_i-\lambda)]^b\\&\times\prod_{i>j=1}^{n}(u_i-u_j)^{-2a/b}\prod_{i>j=1}^{m}(v_i-v_j)^{-2b/a}\prod_{i,j=1}^{n,m}(u_i-v_j)^{-2}\end{aligned} \tag{2.5}$$

where

$$\alpha=\tfrac{1}{3}\left[\frac{a}{b}(n-1)n+\frac{b}{a}(m-1)m-(1+3a)n-(1+3b)m+2nm\right] \tag{2.6}$$

and a, b, n, m are undetermined parameters.

Now, let us consider the behaviour of $J_{AB}(\lambda)$ as $\lambda\to 0$. A simple calculation shows that

$$J_{AB}(\lambda)\to\lambda^{\alpha+\Delta_{AB}} \tag{2.7}$$

and

$$\Delta_{AB}=A(1+2a)+B(1+2b)-\frac{a}{b}A(A-1)-\frac{b}{a}B(B-1)-2AB. \tag{2.8}$$

Since $0 \leqslant A \leqslant n$, $0 \leqslant B \leqslant m$, we know that J_{AB} has $(n+1)(m+1)$ choices. Comparing (2.4) with (2.7), we have three useful equations:

$$N=(n+1)(m+1) \tag{2.9a}$$

$$\frac{a}{b}n(n-1)+\frac{b}{a}m(m-1)-n(1+3a)-m(1+3b)+2nm=\frac{c}{4} \tag{2.9b}$$

$$A(1+2a)+B(1+2b)-\frac{a}{b}A(A-1)-\frac{b}{a}B(B-1)-2AB=2h_i. \tag{2.9c}$$

For RCFT with $L=0$, given the number of characters N, the central term c and conformal scale h_i, we must solve (2.9) and determine a, b, n and m.

Consider, for example, the $SU_k(2)$ WZW model in the diagonal case:

$$C=\frac{3k}{k+2} \qquad h_i=\frac{i(i+1)}{k+2} \qquad \left(i=0,\frac{1}{2},1,\ldots,\frac{k}{2}\right). \tag{2.10}$$

Since h_i depends on one parameter i, the LHS of (2.9c) must have only one parameter. Without any loss of generality, we assume $A=2i$, $B=0$, $b=(1+2k)/2$ and $a=-(1+2k)/4(k+2)$, and can show that they satisfy (2.9) if $n=k$ and $m=0$. In the following sections, we will give the solution of (2.9) for the $SU_k(2)$ WZW and minimal models.

3. The monodromy transformation of the FF integral

Since the modular group of the torus has two generators, it is enough to consider the T and S transformations.

In (2.5), we have chosen the phase so that the integral has zero phase. With the knowledge of the contour integral, the function $J_{AB}(\lambda)$ is canonical at $\lambda=0$, and is defined in the interval $0 \leqslant \lambda \leqslant 1$. In order to discuss the T transformation, we must analytically continue $J_{AB}(\lambda)$ from $0 \leqslant \lambda \leqslant 1$ to $\lambda \leqslant 0$:

$$\begin{aligned}
J_{AB}\left(\frac{\lambda}{\lambda-1}\right) &= [\lambda(\lambda-1)^2]^{\alpha}\int_0^{\lambda/(\lambda-1)}\prod_{i=1}^{A}\mathrm{d}u_i\left[u_i(1-u_i)\left(u_i-\frac{\lambda}{\lambda-1}\right)\right]^a \\
&\times\int_1^{\infty}\prod_{i=A+1}^{n}\mathrm{d}u_i\left[u_i(u_i-1)\left(u_i-\frac{\lambda}{\lambda-1}\right)\right]^a \\
&\times\int_0^{\lambda(\lambda-1)}\prod_{j=1}^{B}\mathrm{d}v_j\left[v_j(1-v_j)\left(v_j-\frac{\lambda}{\lambda-1}\right)\right]^b \\
&\times\int_1^{\infty}\prod_{j=B+1}^{m}\mathrm{d}v_j\left[v_j(v_j-1)\left(v_j-\frac{\lambda}{\lambda-1}\right)\right]^b\prod_{i>j=1}^{n}(u_i-u_j)^{-2a/b} \\
&\times\prod_{i>j=1}^{m}(v_i-v_j)^{-2b/a}\prod_{i,j=1}^{n,m}(u_i-v_j)^{-2}.
\end{aligned} \tag{3.1}$$

Under the transformation T, $J_{AB}(\lambda)$ changes into $J_{AB}(\lambda/(\lambda-1))$. One can show that

$$J_{AB}\left(\frac{\lambda}{\lambda-1}\right)=e^{i\pi(\alpha+\Delta_{AB})}J_{AB}(\lambda). \tag{3.2}$$

From (2.9), we see that $J_{AB}(\lambda)$ has correct transformation properties under the transformation T.

In what follows, we discuss monodromy properties of the FF integral. Generally, under the transformation S, $J_{AB}(\lambda)$ changes as follows:

$$J_{AB}(\lambda)\to J_{AB}(1-\lambda)=\sum_{C,D} S_{AB,CD}J_{CD}(\lambda). \tag{3.3}$$

It is important that the positions of two contours which run through two points can be interchanged without changing the value of the integral. So $S_{AB,CD}$ can be decomposed into two parts: $\alpha_{AC}(n, a, a/b)$ and $\alpha_{BD}(m, b, b/a)$.

In order to calculate $\alpha_{AC}(n, a, a/b)$ we introduce a contour integral:

$$F_A(a,b,\lambda)=\int_0^\lambda \prod_{i=1}^{A} \mathrm{d}u_i[u_i(1-u_i)(\lambda-u_i)]^a \times\int_1^\infty \prod_{i=A+1}^{n} \mathrm{d}u_i[u_i(u_i-1)(u_i-\lambda)]^a \prod_{i>j}^{n}(u_i-u_j)^{-2a/b}. \tag{3.4}$$

Under the monodromy transformation, $F_A(\lambda)$ changes into $F_A(1-\lambda)$, i.e.

$$F_A(\lambda)\to F_A(1-\lambda)=\sum_c \alpha_{AC}\left(n, a, \frac{a}{b}\right)F_c(\lambda). \tag{3.5}$$

According to [1-2], we can write the integration between $(0,\lambda)$ and $(1,\infty)$ as a linear combination in the integrations between $(-\infty, 0)$ and $(\lambda, 1)$. For convenience, we define a contour integral $F(\mu, \nu, \rho, \sigma)$ as follows:

$$\begin{aligned}F(\mu,\nu,\rho,\sigma)=&\int_{-\infty}^0 \prod_{i=1}^{\mu} \mathrm{d}u_i[(1-u_i)(1-u_i)(\lambda-u_i)]^a\int_0^\lambda \prod_{i=\mu+1}^{\mu+\nu} \mathrm{d}u_i[u_i(1-u_i)(\lambda-u_i)]^a\\ &\times\int_\lambda^1 \prod_{i=\mu+\nu+1}^{\mu+\nu+\rho} \mathrm{d}u_i[u_i(1-u_i)(u_i-\lambda)]^a\\ &\times\int_1^\infty \prod_{i=\mu+\nu+\rho+1}^{\mu+\nu+\rho+\sigma} \mathrm{d}u_i[u_i(u_i-1)(u_i-\lambda)]^a\\ &\times\prod_{i>j}^{\mu+\nu+\rho+\sigma}(u_i-u_j)^{-2a/b}.\end{aligned} \tag{3.6}$$

With the help of contour integration, we can obtain two useful formulae:

$$\begin{aligned}F(\mu,\nu,\rho,\sigma)=&\frac{-\mathscr{S}[3a-(2\mu+\mu+\sigma+2\rho-2)a/b]}{\mathscr{S}[2a-(2\rho+\sigma+\nu-1)a/b]}F(\mu+1,\nu-1,\rho,\sigma)\\ &+\frac{-\mathscr{S}[a-(\rho+\sigma)a/b]}{\mathscr{S}[2a-(2\rho+\sigma+\nu-1)a/b]}F(\mu,\nu-1,\rho+1,\sigma)\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}F(\mu,\nu,\rho,\sigma)=&\frac{-\mathscr{S}[a-(\rho+\nu)a/b]}{\mathscr{S}[2a-(\nu+2\rho+\sigma-1)a/b]}F(\mu,\nu,\rho+1,\sigma-1)\\ &+\frac{\mathscr{S}[a-(\mu+\nu)a/b]}{\mathscr{S}[2a-(\nu+2\rho+\sigma-1)a/b]}F(\mu+1,\nu,\rho,\sigma-1)\end{aligned} \tag{3.8}$$

where $\mathscr{S}(x)=\sin(\pi x)$.

We first change all integral contours which run through $(0, \lambda)$ into ones which run through $(-\infty, 0)$ and $(\lambda, 1)$ by using (3.7), then remove away all integral contours over $(1, \infty)$ by using (3.8). Finally, we have

$$\alpha_{k_1 k_1'}\left(n, a, \frac{a}{b}\right) = \sum_{\substack{\mu=1 \\ \mu+\nu=k_1'+1}}^{k_1} \sum_{\nu=1}^{n-k_1+2} \prod_{i=0}^{k_1+\mu+1} \frac{\mathcal{S}[1+3a-2(k_1-2)a/b-(n-k_1-i+1)a/b]}{\mathcal{S}[2a-(n-\mu-3-i)a/b]}$$

$$\times \prod_{i=0}^{\mu-2} \frac{\mathcal{S}[1+a-(n-k_1+i)a/b]}{\mathcal{S}[2a-(\mu+n-k_1-2+i)a/b]}$$

$$\times \prod_{i=0}^{n-k_1-\mu+1} \frac{\mathcal{S}[2+a-(k_1-\mu+i)a/b]}{\mathcal{S}[2a-(n-k_1+2\mu+\nu-4-i)a/b]}$$

$$\times \prod_{i=0}^{\nu-2} \frac{\mathcal{S}[1+a-(\mu-1+i)a/b]}{\mathcal{S}[2a-(2\mu+\nu-4+i)a/b]} \left[\prod_{i=1}^{n-k_1'+1} \mathcal{S}(ia/b) \prod_{i=1}^{k_1'-1} \mathcal{S}(ia/b)\right]$$

$$\times \left[\prod_{i=1}^{k_1-\mu} \mathcal{S}(ia/b) \prod_{i=1}^{\nu-1} \mathcal{S}(ia/b) \prod_{i=1}^{\mu-1} \mathcal{S}(ia/b) \prod_{i=1}^{n-k_1-\nu+2} \mathcal{S}(ia/b)\right]^{-1}. \tag{3.9}$$

So far we have discussed the monodromy transformation in general. In the next section, we consider the modular matrix S in the $SU_k(2)$ WZW model.

4. The modular matrix of the $SU_k(2)$ WZW model

The conformal scale of the $SU_k(2)$ WZW model is

$$h_j = \frac{j(j+1)}{k+2} \tag{4.1}$$

and the central charge $C = 3k/(k+2)$.

As shown in section 2, the FF integral J_{AB} has the same modular invariance as the characters of the $SU_k(2)$ WZW model if we set

$$n=k \qquad m=0 \qquad a=-\frac{1+2k}{4(k+2)}$$
$$b=\frac{1+2k}{2} \qquad B=0 \qquad A=2j \qquad j=0, \frac{1}{2}, \ldots, \frac{k}{2}. \tag{4.2}$$

When q approaches zero, $\chi_j \simeq (2j+1)g^{h_j-c/24}$. So we obtain

$$\chi_j(\lambda) = \frac{2j+1}{N_{2j+1}^{(k+1)}} (16)^{-2}\,[j(j+1)/(k+2) - k/8(k+2)] I_{2j,0}(\lambda) \tag{4.3}$$

where

$$N_{2j+1}^{(k+1)} = \prod_{i=1}^{k-2j} \frac{\Gamma(-ia/b)}{\Gamma(-a/b)} \prod_{i=1}^{2j} \frac{\Gamma(-ia/b)}{\Gamma(-a/b)}$$

$$\times \prod_{i=1}^{k-2j+1} \frac{\Gamma[-\frac{1}{2}-(i+\frac{3}{2})a/b]\Gamma[\frac{1}{2}-(i+\frac{3}{2})a/b]}{\Gamma(1+(4j+3+i)a/b)}$$

$$\times \prod_{i=1}^{2j-1} \frac{\Gamma^2[\frac{1}{2}-(i+\frac{3}{2})a/b]}{\Gamma[1-(2j+i+2)a/b]}. \tag{4.4}$$

Making use of (3.9), we find

$$I_{2j,0}(1-\lambda)=\sum_{j'}\alpha^{k+1}_{2j+1,2j'+1}\left(a,\frac{a}{b}\right)I_{2j',0}(\lambda) \tag{4.5}$$

and

$$\begin{aligned}\alpha^{k+1}_{2j+1,2j'+1}\left(a,\frac{a}{b}\right)=&\sum_{\substack{\mu=1\\ \mu+\nu=2j+2}}^{2i+1}\sum_{\nu=1}^{k-2i+1}\prod_{l=0}^{2i-\mu}\frac{\mathscr{S}[(2i+\frac{1}{2}-l)a/b]}{c[(\mu-1-l)a/b]}\\ &\times\prod_{l=0}^{\mu-2}\frac{\mathscr{S}[(2i-l+\frac{1}{2})a/b]}{c[(\mu-2i-1+l)a/b]}\prod_{l=0}^{k-2j-\nu}\frac{c[(2i+\frac{5}{2}-\mu+l)a/b]}{\mathscr{S}[(2j-2i-1+\mu-l)a/b]}\\ &\times\prod_{l=0}^{\nu-2}\frac{c[(\mu+\frac{1}{2}+l)a/b]}{\mathscr{S}[(2j+1+\mu+l)a/b]}\prod_{l=1}^{k-2i+1-\nu}\mathscr{S}^{-1}(la/b)\prod_{l=1}^{2j}\mathscr{S}(la/b)\\ &\times\prod_{l=1}^{k-2j}\mathscr{S}(la/b)\prod_{l=1}^{2i-\mu+1}\mathscr{S}^{-1}(la/b)\prod_{l=1}^{\nu-1}\mathscr{S}^{-1}(la/b)\prod_{l=1}^{\mu-1}\mathscr{S}^{-1}(la/b)\end{aligned} \tag{4.6}$$

where $c(x)=\cos(x\pi)$.

Comparing (4.6) with (4.1), we get the modular transformation matrix S:

$$S_{ij}=\alpha^{k+1}_{2i+1,2j+1}M_{ji} \tag{4.7}$$

and

$$M_{ij}=\begin{cases}\displaystyle\prod_{l=2i+1}^{2j}\frac{-c[(l+\frac{1}{2})a/b]\mathscr{S}[(l+1)a/b]}{\mathscr{S}[(l+\frac{1}{2})a/b]c[la/b]} & i<j\\ 1 & i=j\\ \displaystyle\prod_{l=2j+1}^{2i}\frac{-c[la/b]\mathscr{S}[(l+\frac{1}{2})a/b]}{\mathscr{S}[(l+1)a/b]c[(l+\frac{1}{2})a/b]} & i>j.\end{cases} \tag{4.8}$$

Equations (3.6) and (3.7) do not appear to be identical to the desired expression for S_{ij} which is given as

$$S_{ij}=\sqrt{\frac{2}{k+2}}\sin\left(\frac{(2i+1)(2j+1)}{k+2}\pi\right). \tag{4.9}$$

The proof of the equality of (4.7) and (4.9) is very difficult. We calculate some simple cases in detail and show their equality in the appendix.

In the $SU_k(2)$ WZW model, there are other modular invariant combinations of the characters, which are D and E series of the A-D-E classification [10]. In FF integral representation, we can write the combinations of characters in the diagonal case (D_{2p+2}, E_6 and E_8) as follows.

For integer $k/4$ ($D_{k/2+2}$), the characters are

$$\hat{\chi}_j=\chi_j+\chi_{k/2-j}\qquad j=0,1,\ldots,\frac{k}{4}. \tag{4.10}$$

One can set

$$\begin{aligned}&n=\frac{k}{4}\qquad m=B=0\qquad a=\frac{2-k}{2(k+2)}\\ &b=\frac{k-2}{4}\qquad A=j=0,1,\ldots,\frac{k}{4}\end{aligned} \tag{4.11}$$

and obtain

$$S_{ij}=\frac{N^{k/4}_{2j+1}(2i+1)}{N^{k/4}_{2i+1}(2j+1)}(16)^{-2}\frac{1}{k+2}[i(i+1)-j(j+1)]\alpha^{k/2+1}_{2i+1,2j+1} \tag{4.12}$$

where

$$\begin{aligned}N^{k/4}_{2j+1}=&\prod_{l=1}^{k/2-2j}\frac{\Gamma(-la/b)}{\Gamma(-a/b)}\prod_{l=0}^{k/2-2j-1}\frac{\Gamma[\frac{1}{2}-(l+\frac{1}{2})a/b]\Gamma[-\frac{3}{2}-(l+\frac{5}{2})a/b]}{\Gamma[1+(4j+1-l)a/b]}\\ &\times\prod_{l=1}^{2j}\frac{\Gamma(-la/b)}{\Gamma(-a/b)}\prod_{l=0}^{2j-1}\frac{\Gamma^2[\frac{1}{2}-(l+\frac{1}{2})a/b]}{\Gamma[1-(2j+l)a/b]}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\alpha^{x/2+1}_{2i+1,2j+1}=&\sum_{\substack{\mu=1\\ \mu+\nu=2j+2}}^{2i+1}\sum_{\nu=1}^{k/2+2i+1}\prod_{l=0}^{2i-\mu}\frac{c[(2i-l+\frac{3}{2})a/b]}{\mathscr{S}[(\mu-l-2)a/b]}\prod_{l=0}^{\mu-2}\frac{c[(2i+\frac{1}{2}-l)a/b]}{-\mathscr{S}[(\mu-2i+l-2)a/b]}\\ &\times\prod_{l=0}^{k/2-2i-\nu}\frac{c[(2i+\frac{3}{2}-\mu+l)a/b]}{\mathscr{S}[(2\mu-l-4+\nu-2i)a/b]}\prod_{l=0}^{\nu-2}\frac{c[(\mu-\frac{1}{2}+l)a/b]}{-\mathscr{S}[(2\mu+\nu-3+l)a/b]}\\ &\times\prod_{l=1}^{2j}\mathscr{S}(la/b)\prod_{l=1}^{k/2-2j}\mathscr{S}(la/b)\prod_{l=1}^{2i-\mu}\mathscr{S}^{-1}(la/b)\prod_{l=1}^{\mu-1}\mathscr{S}^{-1}(la/b)\\ &\times\prod_{l=1}^{\nu-1}\mathscr{S}^{-1}(la/b)\prod^{k/2-2i-\nu+1}\mathscr{S}^{-1}(la/b).\end{aligned} \tag{4.14}$$

For E_6, $k=10$, there are three characters which have conformal scale 0, $\frac{5}{16}$, $\frac{1}{2}$, and central charge $\frac{5}{2}$. We set

$$n=2 \qquad m=B=0 \qquad A=j=0,1,2 \qquad a=-\tfrac{3}{16} \qquad b=-\tfrac{3}{2} \qquad a/b=\tfrac{1}{8} \tag{4.15}$$

and find

$$S_{ij}=\begin{bmatrix}\frac{1}{2} & \sqrt{\frac{1}{2}} & \frac{1}{2}\\ \sqrt{\frac{1}{2}} & \sigma & -\sqrt{\frac{1}{2}}\\ \frac{1}{2} & -\sqrt{\frac{1}{2}} & \frac{1}{2}\end{bmatrix}. \tag{4.16}$$

Here, the formula $\Gamma(2x)=\pi^{1/2}2^{2x-1}\Gamma(\bar{x})\Gamma(x+\frac{1}{2})$ has been used.

For E_8, $k=28$, two characters can be written as

$$\begin{aligned}\chi_1&=[\lambda(1-\lambda)]^{-7/30}(16)^{7/10}\frac{\Gamma(\frac{1}{15})}{\Gamma(-\frac{7}{10})\Gamma(\frac{9}{10})}\int_1^\infty du[u(u-1)(u-\lambda)]^{-1/10}\\ \chi_2&=[\lambda(1-\lambda)]^{-7/30}(16)^{-1/10}7\frac{\Gamma(1+\frac{4}{5})}{\Gamma^2(\frac{9}{10})}\int_0^\lambda du[u(1-u)(\lambda-u)]^{-1/10}.\end{aligned} \tag{4.17}$$

We find the 2×2 matrix S as

$$S=\begin{bmatrix}2/\sqrt{5}\cos 3\pi/10 & 2/\sqrt{5}\cos\pi/10\\ 2/\sqrt{5}\cos\pi/10 & -2/\sqrt{5}\cos 3\pi/10\end{bmatrix}. \tag{4.18}$$

For $D_{2\rho+1}$ and E_7, the characters can be obtained from $A_{4\rho-2}$ and D_{10} with an automorphism of the characters. So the characters of $SU_k(2)$ WZW can be written as FF integrals.

5. The modular transformation for the minimal model

The minimal model is labelled by a pair of coprime integers p and t; the Virasoro central charge C and conformal scale $h_{r,s}$ are

$$c=\frac{1-6(p-t)^2}{pt} \tag{5.1}$$

and

$$h_{r,s}=\frac{(sp-rt)^2-(p+t)^2}{4pt}. \tag{5.2}$$

Since p and t are relatively prime, they cannot be both even. Without any loss of generality, we set p even. The FF representation of characters is as (2.5) with given parameters

$$\begin{aligned} &A=r-1 \qquad 0\leqslant A\leqslant p-2 \qquad B=\frac{(s-1)}{2} \qquad 0\leqslant B\leqslant\frac{(t-3)}{2} \\ &a=\frac{3t-4p}{4p} \qquad b=-\frac{3t-4p}{2t} \qquad n=p-2 \qquad m=\frac{(t-3)}{2}. \end{aligned} \tag{5.3}$$

Now we identify $\chi_{rr'}$ with J_{AB} with (5.3). The normalization is chosen so that $\chi_{rr'}\simeq g^{h_{rr'}-c/24}$ as $q\to 0$. So we can obtain

$$\begin{aligned} \chi_{r,s}=\frac{1}{N_{rs}}16^{-2(h_{rs}-c/24)}\prod_{i=1}^{n+1-r}\frac{\mathcal{S}(a/b)}{\mathcal{S}(ia/b)}\prod_{i=1}^{r-1}\frac{\mathcal{S}(a/b)}{\mathcal{S}(ia/b)} \\ \times\prod_{i=1}^{m+1-s}\frac{\mathcal{S}(b/a)}{\mathcal{S}(ib/a)}\prod_{i=1}^{s-1}\frac{\mathcal{S}(b/a)}{\mathcal{S}(ib/a)}J_{r-1,s-1} \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} N_{r,s}&=j_{n+1-r,m+1-s}\begin{pmatrix}-3a+2(m-2)a/b+2n-4,\ a,\ -a/b\\ -3b+2(n-2)b/a+2m-4,\ b,\ -b/a\end{pmatrix} \\ &\quad\times j_{r-1,s-1}\begin{pmatrix}a,\ b,\ -a/b\\ b,\ a,\ -b/a\end{pmatrix} \\ j_{kl}\begin{pmatrix}\alpha,\beta,\rho\\ \alpha',\beta',\rho'\end{pmatrix}&=\rho^{2kl}\prod_{i,j=1}^{k,l}\frac{1}{(-i+j\rho)}\prod_{i=1}^{k}\frac{\Gamma(i\rho')}{\Gamma(\rho')}\prod_{i=1}^{l}\frac{\Gamma(i\rho)}{\Gamma(\rho)} \\ &\quad\times\prod_{i,j=0}^{k-1,l-1}\frac{1}{(\alpha+j\rho-i)(\beta+j\rho-i)[\alpha+\beta+\rho(l-1-j)-k+1+i]} \\ &\quad\times\prod_{i=0}^{k-1}\frac{\Gamma(1+\alpha'+i\rho')\Gamma(1+\beta'+i\rho')}{\Gamma[2-2l+\alpha'+\beta'+(k-1+i)\rho']} \\ &\quad\times\prod_{i=0}^{l-1}\frac{\Gamma(1+i\rho+\alpha)\Gamma(1+\beta+i\rho)}{\Gamma[2-2k+\alpha+\beta+(l-1+i)\rho]}. \end{aligned} \tag{5.5}$$

Under the transformation S, $\chi_{rs}(\lambda)$ changes into $\chi_{rs}(1-\lambda)$

$$\chi_{rs}(\lambda)\to\chi_{rs}(1-\lambda)=\sum_{r',s'}S_{rs,r's'}\chi_{r's'}(\lambda). \tag{5.6}$$

Making use of (3.9), we can get

$$S_{rs,r's'} = (16)^{2(h_{r's'}-h_{rs})} \frac{N_{r'-1,s'-1}}{N_{r-1,s-1}} \alpha^{n}_{rr'}\left(a, \frac{a}{b}\right) \alpha^{m}_{ss'}\left(b, \frac{b}{a}\right) \tag{5.7}$$

where

$$\begin{aligned}
\alpha^{n}_{rr'}\left(a, \frac{a}{b}\right) = & \sum_{\substack{\mu=1 \\ \mu+\nu=r'+1}}^{r} \sum_{\nu=1}^{n-r+1} \prod_{i=0}^{r-\mu-1} \\
& \times \frac{\mathscr{S}[1+3a-2(r-2)a/b-(n-r-i)a/b]}{\mathscr{S}[2a-(n+\mu-3-i)a/b]} \\
& \times \prod_{i=0}^{\mu-2} \frac{\mathscr{S}[1+a-(n-r+i)a/b]}{\mathscr{S}[2a-(\mu+n-r-2+i)a/b]} \\
& \times \prod_{i=0}^{n-r-\nu} \frac{\mathscr{S}[2+a-(r-\mu+i)a/b]}{\mathscr{S}[2a-(n-r+2\mu+\nu-4+i)a/b]} \\
& \times \prod_{i=0}^{\nu-2} \frac{\mathscr{S}[1+a-(\mu-1+i)a/b]}{\mathscr{S}[2a-(2\mu+\nu-4+i)a/b]} \\
& \times \left[\prod_{i=1}^{r'-1} \mathscr{S}(ia/b) \prod_{i=1}^{n-r'} \mathscr{S}(ia/b)\right] \\
& \times \left[\prod_{i=1}^{n-r-\nu+1} \mathscr{S}(ia/b) \prod_{i=1}^{\nu-1} \mathscr{S}(ia/b) \prod_{i=1}^{\mu-1} \mathscr{S}(ia/b) \prod_{i=1}^{r-\mu} \mathscr{S}(ia/b)\right]^{-1} .
\end{aligned} \tag{5.8}$$

This is the (A_{p-1}, A_{t-1}) series of the minimal model classification. As discussed in the above section, the characters of $(D_{2+(p-1)/2}, A_{t-1})$ can be written as FF integrals. Here, we only give the result for this case:

$$\begin{aligned}
& n = \frac{(p-2)}{4} \qquad m = \frac{(t-3)}{2} \qquad A = \frac{r=1}{2} \qquad B = s-1 \\
& a = \frac{4t-3p}{2p} \qquad b = -\frac{4t-3p}{4t}
\end{aligned} \tag{5.9}$$

and

$$S_{rs,r's'} = (16)^{2(h_{r's'}-h_{rs})} \frac{N_{r'-1,s'-1}}{N_{r-1,s-1}} \alpha^{n}_{rr'}\left(a, \frac{a}{b}\right) \alpha^{m}_{ss'}\left(b, \frac{b}{a}\right). \tag{5.10}$$

Generally, for $(D_{1+(p-1)/2}, A_{t-1})$ and (B_k, A_{t-1}) $(k=6, 7, 8)$ the characters cannot be directly written as the FF contour integrals, but they can be written as a linear combination of the A-series characters. So the characters of the $SU_k(2)$ WZW and minimal models can be represented as the FF integrals.

6. Conclusion

In this paper, we have discussed the modular property of characters of RCFT in FF contour integral representation and given explicitly the expression of modular transformation matrices. In fact, all characters whose Wronskian determinant is zero can be written as an FF integral. So the FF integral gives a sign to construct characters of conformal field theories (at least ones of rational conformal field theories). This can be generalized to construct the characters of non-unitary $SU_k(2)$ WZW [11] and other models.

Acknowledgment

We are grateful to Professors K J Shi and P Wang for useful discussion. This work was supported in part by the National Science Foundation of China through the Nankai Institute of Mathematics.

Appendix

Here we shall describe the calculation of S_{ij} in some simple cases. For the $SU_k(2)$ WZW model, we obtain S_{ij} from (5.6), (5.7) and (5.8), which can be written as follows:

$$S_{ij} = M_{ji}\alpha^{k+1}_{2i+1,2j+1} \tag{A1}$$

and

$$\alpha^{k+1}_{2i+1,2j+1} = \sum_{\substack{\mu=1 \\ \mu+\nu=2j+2}}^{2i+1} \sum_{\nu=1}^{k-2i+1} \prod_{l=0}^{2i-\mu} \frac{S(2i+\frac{1}{2}-l)}{-C(\mu-1-l)} \prod_{l=0}^{\mu-2} \frac{S(2i+\frac{1}{2}-l)}{-C(\mu-2i-1+l)}$$

$$\times \prod_{l=0}^{k-2i-\nu} \frac{c(2i+\frac{5}{2}-\mu+l)}{-s(2j-2i-1+\mu-l)} \prod_{l=0}^{\nu-2} \frac{c(\mu+\frac{1}{2}+l)}{-s(2j+1+\mu+l)}$$

$$\times \prod_{l=1}^{2j} s(l) \prod_{l=1}^{k-2j} s(l) \prod_{l=1}^{2i-\mu+1} s^{-1}(l) \prod_{l=1}^{\nu-1} s^{-1}(l) \prod_{l=1}^{\mu-1} s^{-1}(l) \prod_{l=1}^{k-2i-\nu+1} s^{-1}(l) \tag{A2}$$

where $s(t) = \sin(t\pi/2(k+2))$, $c(t) = \cos(t\pi/2(k+2)$.

First, setting $j = k/2$ we have

$$S_{ik/2} = \prod_{l=1}^{k-2i} \frac{c(l+1)c(l+\frac{1}{2}+2i)}{c(2i+l-1)s(l)} \prod_{l=1}^{2i} \frac{s(l-2i-\frac{3}{2})}{c(l-1)} \prod_{l=2i+1}^{k} \frac{c(l)s(l+\frac{1}{2})}{c(l+\frac{1}{2})s(l+1)}. \tag{A3}$$

Making use of the formula

$$\prod_{r=1}^{n-1} \sin\frac{\pi r}{n} = n2^{-(n-1)} \tag{A4}$$

one can show that $S_{ik/2}$ in (A3) is equal to $\sqrt{2/(k+2)}\sin(k+1)(2i+1)\pi/(k+2)$.

Secondly, setting $j = 0$, we can show

$$S_{i0} = \prod_{l=1}^{k} s(l+\tfrac{1}{2}) \prod_{l=1}^{k+1} \frac{1}{s(l)} \times s(l)s(4i+2)$$

$$= \sqrt{\frac{2}{k+2}}\sin\frac{2i+1}{k+2}\pi. \tag{A5}$$

Similarly

$$S_{0j} = \prod_{l=1}^{k-1} \frac{1}{c(l)} \prod_{l=1}^{k-} s(l+\tfrac{1}{2}) \frac{s(2j+1)c(2j+1)}{c(1)s(1)} = \sqrt{\frac{2}{k+2}}\sin\frac{2j+1}{k+2}\pi. \tag{A6}$$

For $i=\frac{1}{2}$, we find the following identity from (A2):

$$\alpha^{k+1}_{2,2j+1}=-\frac{s(\frac{3}{2})}{s(1)}\prod_{l=1}^{k-1-2j}\frac{c(\frac{3}{2}+l)}{c(2j-l)}\prod_{l=1}^{2j-1}\frac{c(l+\frac{3}{2})}{-s(2j+2+l)}\left[-\frac{c(\frac{3}{2})c(2j+2)}{s(2j+2)}+\frac{c(\frac{3}{2})s(2j)}{c(2j)}\right]$$

$$=\frac{1}{2}\frac{s(3)c(4j+2)}{c(2j)s(1)}\prod_{l=1}^{k-2j}c(l+\tfrac{1}{2})\prod_{l=1}^{2j}c(l+\tfrac{1}{2})\prod_{l=1}^{2j-1}\frac{1}{c(l)}$$

$$\times\prod_{l=1}^{k-2j-1}\frac{1}{c(l)}\prod_{l=2j+2}^{4j+1}\frac{1}{s(l)}. \tag{A7}$$

By a direct calculation, we have shown that

$$S_{1/2j}=M_{j1/2}\,\alpha^{k+1}_{2,2j+1}=\sqrt{\frac{2}{k+2}}\sin\pi\frac{2(2j+1)}{k+2} \tag{A8}$$

For small k, one can directly calculate S_{ij} and find

$$S_{ij}=\sqrt{\frac{2}{k+2}}\sin\frac{(2i+1)(2j+1)}{k+2}\,\pi. \tag{A9}$$

We have to point out that for an arbitrary integer k, the proof of (A9) is difficult. For a given k, one can find that (A9) is true by a tedious calculation.

References

[1] Dotsenko S Vl and Fateev V A 1984 *Nucl. Phys.* B **240** 312
[2] Dotsenko S Vl and Fateev V A 1985 *Nucl. Phys.* B **251** 691
[3] Felder G 1988 *Nucl. Phys.* B **317** 215
[4] Felder G, Frohlich J and Keller G 1989 *Commun. Math. Phys.* **124** 647
[5] Hou B Y, Lie D P and Yue R H 1989 *Phys. Lett.* B **229** 45
[6] Hou B Y, Shi K J, Wang P and Yue R H 1989 *Preprint NWU-IMP-1219*
[7] Mathur S D, Mukhi S and Sen A 1988 *Phys. Lett.* B **213** 303
[8] Mathur S D, Mukhi S and Sen A 1989 *Nucl. Phys.* B **312** 15
[9] Mukhi S, Panda S and Sen A 1989 *Preprint TIFR/TH/89-01*
[10] Capelli A, Itzykson C and Zuber J B 1987 *Nucl. Phys.* B **280** 445; 1987 *Commun. Math. Phys.* **113** 1
[11] Mukhi S and Panda S 1989 *Preprint TIFR/TH/89-64*